

ON SUBSETS OF A CONTINUOUS CURVE WHICH LIE ON AN ARC OF THE CONTINUOUS CURVE

A Dissertation

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE
UNIVERSITY OF MICHIGAN

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EDWIN W. MILLER

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ON SUBSETS OF A CONTINUOUS CURVE WHICH LIE ON AN ARC OF THE CONTINUOUS CURVE.*

By EDWIN W. MILLER

Introduction. In 1919 R. L. Moore and J. R. Kline gave the following Theorem: †

If M is a closed and bounded plane point set, in order that M be a subset of an arc (of the plane) it is necessary and sufficient that the following two conditions be satisfied:

1. Every component of M is an arc or a point.
2. No interior point of an arc-component, t , of M is a limit point of $M - t$.

As the plane is a particular example of a continuous curve, ‡ the problem considered by Moore and Kline appears as a special case of the following general problem:

If S is a continuous curve in n -dimensional Euclidian space, E_n , and M is a closed and bounded subset of S , what conditions are necessary and sufficient that M be a subset of an arc of S ?

It is with this problem that the present paper is chiefly concerned. The restriction that M be closed is not a vital one, as any set M is a subset of an arc if and only if the same is true of its closure $\bar{M}$.

Part I is given over to a study of the Moore-Kline conditions, which are of exceptional interest because of their purely internal character; that is to say, these conditions may be defined without reference to the imbedding space. As the proof of the above mentioned theorem of Moore and Kline makes use of properties of E_2 which are not properties of E_n ($n > 2$) a different procedure from that pursued by Moore and Kline is adopted to obtain the result that the Moore-Kline theorem still holds if M is any bounded and closed subset of E_n ($n > 2$).

In Part II a theorem is given which reduces the general problem for any continuous curve in E_n to the following problem: If a and b are two

* The chief results of Part I of this paper were presented under separate title to the American Mathematical Society on November 30, 1928. Part II was presented to the Society on April 30, 1930.

† *Annals of Mathematics*, Vol. 20 (1919), pp. 218-223.

‡ Throughout this paper the term "continuous curve" is used to mean a locally connected (connected in kleinen) continuum.

points of a cyclicly connected* continuous curve S , and M is a closed subset of S , under what conditions is M a subset of an arc in S whose end points are a and b ? This problem remains unsolved as far as the present paper is concerned. However, with the aid of the above mentioned theorem complete solutions of the problem are obtained for certain important classes of continuous curves.

The writer wishes to acknowledge his indebtedness to Professor R. L. Wilder, who suggested the problem of this paper, and who aided greatly in its preparation by his criticism and advice.

I.

THEOREM 1. *In E_n ($n > 2$) in order that the closed and bounded set M be a subset of an arc it is necessary and sufficient that*

1. *Each component of M is either an arc or a point.*
2. *If t is an arc-component of M , then no interior point of t is a limit point of $M - t$.*

Proof. The conditions are evidently necessary.

They are also sufficient. L. W. Cohen has shown † that any set M in a separable metric space is homeomorphic with a subset of the linear continuum if and only if four conditions are satisfied, two of which conditions are the conditions given above, and the remaining two of which are:

3. If p is a point-component of M then $\dim_p M = 0$; ‡ and
4. If a is an end point of an arc-component t of M then $\dim_a (M - t + a) = 0$.

It will be shown that for a closed and bounded set M in E_n (the particular separable metric space here considered) condition (3) is satisfied and that (4) follows from (2). In fact, let F be a closed and bounded set in E_n , K a component of F , ϵ any positive number and H the set of all points of F which are at a distance $\geq \epsilon$ from K . The set H , as is readily shown, is closed. It follows from a theorem of Miss Anna Mullikin § that F is the sum of two mutually exclusive closed sets F_1 and F_2 such that $F_1 \supset K$ and $F_2 \supset H$. If K is a single point we have the result that a closed set in E_n is 0-dimensional at any one of its point-components. Thus (3) is at once seen to be satisfied. From (2) it follows that if t is any arc-component of M and a and b are the end points of t , then $M - t + a + b$ is a closed set. The

* A continuous curve S is *cyclicly connected* if every two points of S lie on some simple closed curve which is a subset of S .

† *Fundamenta Mathematicae*, Vol. 14 (1929), pp. 281-303.

‡ For definitions of dimension, see Menger, *Dimensionstheorie* (Leipzig, 1928).

§ *Transactions of the American Mathematical Society*, Vol. 24 (1922), pp. 144-162.

point a is evidently a point-component of $M - t + a + b$. Hence $\dim_a(M - t + a + b) = 0$. But $\dim_a(M - t + a + b) = \dim_a(M - t + a)$. Thus (4) is also satisfied. The set M , then, is homeomorphic with some subset $\bar{M}$ of the linear continuum. Since M is closed and bounded, $\bar{M}$ is a subset of some linear interval I . Let $\bar{a}$ be the first point of $\bar{M}$ on I and $\bar{b}$ the last point of M on I , and consider the interval $(\bar{a}, \bar{b})$. Let us arrange the open intervals complementary to $\bar{M}$ in a sequence $(\bar{a}_i, \bar{b}_i)$. It will be shown that corresponding to every i there is an arc N_i in E_n whose end points, a_i and b_i , are the points of M which correspond to $\bar{a}_i$ and $\bar{b}_i$, and which is such that $W = M + \sum_{i=1}^{\infty} N_i$ is an arc from a to b , the points of M corresponding to $\bar{a}$ and $\bar{b}$.

For this purpose we shall extend the following

*Theorem of J. R. Kline:** In E_n ($n \geq 2$) if H is the sum of a countable infinity of closed sets M_i , such that no $M_i = E_n$, and no M_i disconnects any domain, then if D is a domain, $D - D \cdot H$ is arc-wise connected.

THEOREM A. Under the conditions of the above theorem if a and b are any two points of D , there is an arc ab such that $ab - a - b \subset D - D \cdot H$.

Proof. If $a \in D \cdot H$, then a is a limit point of $D - D \cdot H$. In fact, since the closed set $M_i \neq E_n$, and disconnects no domain of E_n , it is easily shown to be nowhere dense. Hence H is of the first category of Baire, so that a is a limit point of $D - D \cdot H$. We can evidently find a sequence of distinct points $\{a_n\}$ such that $\lim a_n = a$, and such that the sphere

$$S_n = S(a_n, 2d(a_n, a_{n+1}))$$

is a sub-domain of D . By the above theorem of Kline, $S_n - S_n \cdot H$ contains an arc $a_n a_{n+1}$. The point set $a + \sum_{n=1}^{\infty} a_n a_{n+1}$ is evidently a continuous curve containing a_1 and a . There is, then, an arc $a_1 a \subset a + \sum_{n=1}^{\infty} a_n a_{n+1}$ (R. L. Moore)†. Evidently, except for the point a , the arc $a_1 a \subset D - D \cdot H$. From this result Theorem A follows readily.

Now, the set T consisting of all point-components of M together with the end points of all arc-components is clearly a closed, totally disconnected set. Furthermore, as Cohen shows,‡ the arc-components of M are countable.

* *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 290-292. The conditions here imposed on H are those given by R. L. Wilder, *ibid.*, Vol. 23 (1927), p. 388.

† See, for instance, R. L. Moore, *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 233-236.

‡ *Loc. cit.*

As any closed totally disconnected set is 0-dimensional and any arc is 1-dimensional,* M is the sum of a countable family of closed sets each of dimension ≤ 1 . The set $M = T + \sum_{i=1}^{\infty} M_i$. Urysohn has shown † that a closed set of dimension $\leq n-2$ disconnects no domain of E_n . Since we are here concerned with $E_n (n \geq 3)$, neither T nor any one of the arc-components M_i disconnects any domain of E_n . Evidently neither T nor any one of the sets $M_i = E_n$. Hence the conditions of Theorem A are satisfied. Thus, if $S_1 = S(a_1, 2d(a_1, b_1))$ there is an arc N_1 from a_1 to b_1 such that

$$N_1 - a_1 - b_1 \subset S_1 - S_1 \cdot M.$$

The set $M + N_1$ satisfies the conditions of Theorem A. Thus, if

$$S_2 = S(a_2, 2d(a_2, b_2))$$

there is an arc N_2 from a_2 to b_2 such that

$$N_2 - a_2 - b_2 \subset S_2 - S_2 \cdot (M + N_1).$$

In general, the set $M + \sum_{i=1}^{n-1} N_i$ satisfies the conditions of Theorem A. Hence if $S_n = S(a_n, 2d(a_n, b_n))$ there is an arc N_n from a_n to b_n such that

$$N_n - a_n - b_n \subset S_n - S_n \cdot (M + \sum_{i=1}^{n-1} N_i).$$

It will now be shown that $W = M + \sum_{i=1}^{\infty} N_i$ is an arc.

As $\bar{M}$ is a closed bounded set the mapping of $\bar{M}$ on to M is uniformly continuous. Since the intervals $(\bar{a}_n, \bar{b}_n)$ do not overlap and all lie on a finite interval, $\lim_n d(\bar{a}_n, \bar{b}_n) = 0$. It follows that $\lim_n d(a_n, b_n) = 0$. Therefore $\lim_n \text{diam}(S_n) = 0$ and as $N_n \subset S_n$, $\lim_n \text{diam}(N_n) = 0$. Now let p be a limit point of W . If p is a limit point of M then $p \in W$. If p is a limit point of $\sum_{i=1}^{\infty} N_i$, it either belongs to a particular N_i and therefore to W , or else it is a limit point of a set of points only a finite number of which belong to any given N_i . In this case, since $\lim_n \text{diam}(N_n) = 0$, it is a limit point of the end points of the arcs N_i . But the end points of N_i are points of M . Thus W is closed.

If W is not connected it is the sum of two mutually exclusive non-vacuous closed sets Q_1 and Q_2 . Not all the points of M belong to Q_1 (for example)

* See Menger, *loc. cit.*

† *Fundamenta Mathematicae*, Vol. 8 (1926), p. 311.

for then we would have $Q_2 = 0$. Let us put, then, $M \cdot Q_1 = M'$ and $M \cdot Q_2 = M''$. The set $M = M' + M''$ where M' and M'' are both closed non-vacuous sets. This separation of M determines a separation of $\bar{M}$ into two mutually exclusive closed non-vacuous sets $\bar{M}'$ and $\bar{M}''$. Let $\bar{p}$ be any point of $\bar{M}'$. As $\bar{p}$ is not a limit point of $\bar{M}''$, and as there is at least one point of $\bar{M}''$ either to the left of $\bar{p}$ or to the right,—say to the right, there will be a first point $\bar{q}$ of the closed set $\bar{M}''$ to the right of $\bar{p}$. Since $\bar{q}$ is not a limit point of $\bar{M}'$ and since there is at least one point of $\bar{M}'$ to the left of $\bar{q}$, viz. $\bar{p}$ —there is a last point $\bar{p}'$ of the closed set $\bar{M}'$ to the left of $\bar{q}$. Now $\bar{p}'$ and $\bar{q}$ are the end points of an interval complementary to $\bar{M}$. Therefore there is an arc N_i whose end points are p' and q . Hence $p' + q \subset Q_1$, or else $p' + q \subset Q_2$. It follows that $\bar{p}' + \bar{q} \subset \bar{M}'$ or else $\bar{p}' + \bar{q} \subset \bar{M}''$. This is in contradiction with the way in which $\bar{p}'$ and $\bar{q}$ were chosen.

We now show that $W - p$ is not connected if $W - (a + b) \supset p$.

Case 1. The point p belongs to M .

Let $W - p = W_1 + W_2$, where W_1 consists of all points x of M such that $\bar{x}$ is to the left of $\bar{p}$, together with all points x of W which are interior points of arcs N_i such that $\bar{a}_i$ and $\bar{b}_i$ are not to the right of $\bar{p}$; and where $W_2 = (W - p) - W_1$. The set $W_1 \neq 0$ since $W_1 \supset a$, and $W_2 \neq 0$ since $W_2 \supset b$. Now, let $q_1 \in W_1$. There exists a sphere $S(q_1, d)$ such that $S(q_1, d) \cdot M \cdot W_2 = 0$ since the only limit point of the points of $\bar{M}$ to the right of $\bar{p}$ which is not a point of this set is the point $\bar{p}$. Since $\lim_i \text{diam}(N_i) = 0$, there is a positive integer r such that if $i > r$, then $\text{diam}(N_i) < d/2$. Hence if $i > r$, and if $N_i \subset W_2 + p$, then $S(q_1, d/2) \cdot N_i = 0$. Let F consist of all points z such that $z \in N_i$ where $i \leq r$. The set F is closed and $F \cdot q_1 = 0$. Since q_1 is neither a limit point of $M \cdot W_2$ nor of $W_2 \cdot \sum_{i=1}^{\infty} N_i$, it is not a limit point of W_2 . Similarly no point of W_2 is a limit point of W_1 .

Case 2. p is not a point of M .

Then p is a point of some N_i ,—say N_k . If $a_k \neq a$, and $b_k \neq b$, then a_k and b_k determine separations: $W - a_k = W_1^{(a_k)} + W_2^{(a_k)}$, and $W - b_k = W_1^{(b_k)} + W_2^{(b_k)}$ in virtue of the result of Case 1. Denote by $a_k p$ the arc of N_k from a_k to p and by $p b_k$ the arc of N_k from p to b_k , and put $W - p = K_1 + K_2$; where $K_1 = (W_1^{(a_k)} + a_k p - p)$ and $K_2 = (p b_k - p + W_2^{(b_k)})$. Since $K_1 \subset W_1^{(b_k)}$ it contains no limit point of $W_2^{(b_k)}$. It evidently contains no limit point of $p b_k - p$. Since $K_2 \subset W_2^{(a_k)}$ it contains no limit point of $W_1^{(a_k)}$. It evidently contains no limit point of $a_k p - p$. Thus $W - p$ is the sum of two mutually separated sets. In case $a_k = a$ or $b_k = b$, we can effect

a separation of $W - p$ in a similar fashion. This concludes the proof that W is an arc.

From Theorem 1 and the Moore-Kline theorem we have at once the following

COROLLARY: *If $M \subset E_n$ and $E_n \subset E_m$, where E_n and E_m are Euclidian spaces of n and m dimensions, then if M is a subset of an arc in E_m it is also a subset of an arc in E_n .*

THEOREM 2. *If M is a closed and bounded subset of a (connected) domain D of E_n , and if M satisfies the conditions of Moore and Kline, then there is an arc A such that $M \subset A \subset D$.*

Proof. By Theorem 1 there is an arc A' in E_n which contains M . If $q \in A' \cdot D$, and C_q is the component of $A' \cdot D$ determined by q , only a finite number of distinct components C_q can contain points of M . For suppose there is an infinity of such components — $C_1, C_2, \dots, C_n, \dots$, and let $a_n \in C_n \cdot M$. The points a_n are all distinct, and since M is bounded, the set $[a_n]$ has at least one limit point, p . Since M is closed, $p \in M$ and therefore $p \in D$. The component C_p is evidently a closed, half-closed or open arc, t .*

If t is a closed arc then evidently $A' \subset D$. If t is a half-closed arc denote $t +$ both its end points by t' . That end point a of t' which belongs to t is evidently an end point of the arc A' . In this case, then, p is either an end point of A' or an inner point of t' . But, clearly, then, since A' is an arc, p is not a limit point of $A' - t'$ and therefore not of $[a_n]$. Finally, if t is an open arc, and t' denotes $t +$ its end points, then p is an inner point of t' . But then, p is not a limit point of $A' - t'$, and therefore is not a limit point of $[a_n]$. Hence only a finite number of components C_q of $A' \cdot D$ contain points of M . Since M is a closed subset of D it is apparent that each C_q contains a closed sub-arc containing all the points of $M \cdot C_q$. Thus there exists a finite number of mutually exclusive arcs $A_1, \dots, A_m$ such that $M \subset \sum_{i=1}^m A_i \subset D$. It is an easy consequence of the Wilder accessibility theorem † that in E_2 the finite set of arcs $\sum_{i=1}^m A_i$ is contained in an arc of D . For E_n with $n > 2$ successive application of Theorem A leads to a like result.

THEOREM 3. *If D is a domain of E_n , and M is a closed bounded subset*

* An arc is said to be closed if it contains both end points, half-closed if it contains just one end point, and open if it contains neither end point.

† R. L. Wilder, *Fundamenta Mathematicae*, Vol. 7 (1923), pp. 340-377.

of D which satisfies the Moore-Kline conditions, and if p_1 and p_2 are distinct points of M , then there is an arc p_1p_2 such that $M \subset p_1p_2 \subset D$ if and only if.

(1) p_1 and p_2 are point-components of M ; or

(2) p_1 and p_2 are end points of distinct arc-components t_1 and t_2 of M such that p_1 is not a limit point of $M - t_1$ and p_2 is not a limit point of $M - t_2$; or

(3) one of the points p_1, p_2 is a point-component of M and the other is an end point of an arc-component t of M , and is not a limit point of $M - t$.

Proof. Since no interior point of an arc-component t of M can be an end point of an arc which contains M , and since the same is true of an end point of an arc-component t of M which is a limit point of $M - t$, the conditions are at once seen to be necessary.

We now show that the conditions are also sufficient. By Theorem 2 there is an arc A such that $M \subset A \subset D$. We may suppose that the end points a_1 and a_2 of A are points of M . Let us first assume that neither a_1 nor a_2 is p_1 or p_2 , and consider the case $n = 2$. On the basis of well-known results it is easy to show that there exists a simple closed curve J such that if I is the interior of J , then $J + I \subset D$ and $A \subset J$. If either p_1 or p_2 is an end point of an arc-component t of M and is not a limit point of $M - t$, or else is a point-component of M and is not a limit point of M from both sides on A , the simple closed curve J evidently contains an arc A' which contains M and has one of the two points, p_1 and p_2 ,—let us say, p_1 , as an end point. If both of the points p_1 and p_2 are point-components of M and limit points of M from both sides on A , we shall construct an arc A' which contains M and has p_1 as one of its end points as follows: We may evidently suppose that the order of the points a_1, p_1, p_2 , and a_2 is a_1, p_1, p_2, a_2 . Let us choose two sequences of sub-arcs of A complementary to M , with end points x_n and y_n and x'_n and y'_n respectively, so that

$$a_1 < x'_{n-1} < y'_{n-1} < x'_n < y'_n < p_1 < x_n < y_n < x_{n-1} < y_{n-1} < p_2,$$

and such that $\lim x'_n = \lim y'_n = \lim x_n = \lim y_n = p_1$.

There is an arc α_1 , from x'_1 to x_1 such that $\alpha_1 - x'_1 - x_1 \subset I$. The arc α_1 forms with the sub-arc x'_1x_1 of A a simple closed curve J_1 whose interior $I_1 \subset I$. There is an arc α'_1 from y_2 to y'_1 such that $\alpha'_1 - y_2 - y'_1 \subset I_1$. The arc α'_1 forms with the sub-arc $y_2y'_1$ of A a simple closed curve J'_1 whose interior $I'_1 \subset I_1$.

We proceed in this way to construct arcs α_2 from x'_2 to x_2 , α'_2 from y_3 to y'_2 , α_3 from x'_3 to x_3 , and so on. Now, it is possible to choose the arcs α_n

and α'_n so that $\lim \text{diam}(\alpha_n) = \lim \text{diam}(\alpha'_n) = 0$. In fact, let us prove the following

LEMMA. *If x and y are two points of a simple closed curve J , whose interior is I , and xy is either arc of J from x to y , then there exists an arc α from x to y such that $\alpha - x - y \subset I$, and $\text{diam}(\alpha) < \text{diam}(xy) + \epsilon$, where ϵ is a preassigned positive number.*

Proof. Let D denote the set of points of I at a distance $< \epsilon/2$ from the arc xy . It is easily shown that D is an open and connected subset of I . The points x and y lie on the boundary of D and neither belongs to any continuum of condensation of the boundary. Hence, by the Wilder accessibility theorem, x and y are both accessible from D . It follows at once that there is an arc α in I from x to y such that $\text{diam}(\alpha) < \text{diam}(xy) + \epsilon$.

We will suppose, then, that the arcs α_n and α'_n are chosen so that $\lim \text{diam}(\alpha_n) = \lim \text{diam}(\alpha'_n) = 0$. Then the point set A' , consisting of the arc $y_1 p_2 a_2$ of A , that arc $x'_1 a_2$ of J which does not contain p_1 , the arcs α_n and α'_n , and the sub-arcs $x_n y_{n+1}$ and $y'_n x'_{n+1}$ of A ($n = 1, 2, \dots$) is, as can easily be shown, an arc from p_1 to y_1 which contains M and lies in D .

An argument which is only a slight modification of the one given above suffices to show that starting with the arc A' we can construct an arc A'' of D which contains M and has p_1 and p_2 as its end points. This completes the proof for the case $n = 2$.

For the case $n > 2$, our procedure is similar. By Theorem A, the end points of the arc A are accessible from D . There is then a simple closed curve J such that $A \subset J \subset D$. As in the case $n = 2$, we are reduced to the supposition that p_1 is a point-component of M which is a limit point of M from both sides on A . We choose two sequences of sub-arcs of A complementary to M as before, and construct arcs α_n and α'_n , all mutually exclusive, such that $\lim \text{diam}(\alpha_n) = \lim \text{diam}(\alpha'_n) = 0$. That there exist two such sequences of arcs α_n and α'_n is readily seen to be a consequence of Theorem A. The rest of the proof runs parallel to the proof for the case $n = 2$.

It is natural, in this connection, to consider the problem of determining conditions under which a closed point set M in E_n is a subset of a ray or an open curve.*

THEOREM 4. *A closed set M in E_n is a subset of a ray if and only if the two following conditions are satisfied:*

1. *Either (a) the components of M are all arcs and points, or (b) the*

* For definitions of these, see R. L. Moore, *Transactions of the American Mathematical Society*, Vol. 21 (1920), p. 347.

components of M are a single ray and a bounded collection of arcs and points; and

(2) No interior point of an arc- or ray-component, t , is a limit point of $M - t$.

The conditions are evidently necessary. Their sufficiency is proved as follows: Suppose, first, that 1(a) and 2 hold. Let $p \in E_n - M$, and let $S = S(p, d)$ be such that $S \cdot M = 0$. If S is taken as the sphere of inversion, the set M maps into a set $M' \subset S$. By familiar properties of the transformation of inversion, the set $M' + p$ is closed and satisfies the Moore-Kline conditions. The point p is a point-component of $M' + p$. Thus, by Theorem 3 there is an arc which contains $M' + p$, lies in S , and has p as one of its end points. The transform of this arc is a ray which contains M .

If 1(b) and 2 hold, $M' + p$ is again a set satisfying the Moore-Kline conditions. In this case p is an end point of an arc-component t of $M' + p$ and is not a limit point of $(M' + p) - t$. Again, Theorem 3 gives us the existence of an arc in S which contains $M' + p$ and has p as one end point. The transform of this arc is a ray which contains M .

It is here to be noted that G. T. Whyburn has given conditions which are necessary and sufficient that a closed set in E_2 be a subset of an open curve.* Whyburn's proof depends upon the Moore-Kline theorem. Using Theorem 1 of this paper in place of the Moore-Kline theorem, the result can be extended to $E_n (n > 2)$.

II.

In the first part of this section it will be shown that the Moore-Kline conditions constitute a solution of the problem of this paper only for a rather limited class of continuous curves.

THEOREM 5. *If S is a plane closed set, and if every one of its bounded closed subsets which satisfies the Moore-Kline conditions is a subset of an arc in S , then S is either the whole plane or a simple continuous curve.*

Proof. To abbreviate we will use the expression "a set E " to mean any bounded closed subset of S which satisfies the Moore-Kline conditions.

Since any pair of points of S is a set E , S is arc-wise connected and therefore connected. Since S is closed, it is a continuum. S is, in fact, a continuous curve. For if S is not a continuous curve, then in virtue of a

* *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 746-754; Theorem 5.

theorem of R. L. Wilder's* there is a point p of S and a circular neighborhood K of p such that p is a sequential limit point of a set of points $[p_n]$ of S each of which lies in a different quasi-component of $S \cdot K$. Evidently, $p + \sum p_n$ is a set E , and if t were an arc of S containing it, there would be in $S \cdot K$ a sub-arc t' of t such that $t' \supset p + \sum_{n=k}^{\infty} p_n (k \geq 1)$. This, of course, contradicts the fact that no two points p_k and p_j lie in the same component of $S \cdot K$, and consequently S is a continuous curve.

It will now be shown that S is either the whole plane or else a simple continuous curve. Let us suppose, first, that S contains no simple closed curve. If S contained a point p which separated S into three mutually separated sets, S_1 , S_2 , and S_3 , then if $p_i \in S_i$ ($i = 1, 2, 3$), the set $\sum_{i=1}^3 p_i$ would be a set E which is clearly not a subset of any arc in S . Hence, no point p of S separates S into more than two components.

If S is bounded, it has at least two non-cut points, q_1 and q_2 .† Let t be an arc of S from q_1 to q_2 . If $t \neq S$, there is a component C of $S - t$ which has—since S is an acyclic continuous curve—a unique limit point q_3 in t . The set $S - q_3$, as is easily shown, would consist of three mutually separated sets. We have seen that this is impossible. Hence, if S is bounded, it is an arc.

Let us now suppose that S is unbounded and that it contains a non-cut point, p . The point p is the end point of a ray r of S .‡ If $r \neq S$, the unique limit point in r of a component C of $S - r$ would break up S into three mutually separated sets. Hence if S is unbounded and contains a non-cut point, it is a ray.

Let us suppose, finally, that S is unbounded and is disconnected by each one of its points. Thus if $p \in S$, the set $S - p$ consists of two mutually separated connected sets, S_1 and S_2 . The point p is a non-cut point of $S_i + p$ ($i = 1, 2$), and by the argument just employed $S_i + p$ is a ray r_i , so that $S = r_1 + r_2$ is an open curve.

Now, let us suppose that S does contain a simple closed curve, J . Then S is cyclically connected. For, suppose not. Then J determines a maximal

* *Proceedings of the National Academy of Sciences*, Vol. 15 (1929), pp. 614-621; Theorem 1.

† S. Mazurkiewicz, *Fundamenta Mathematicae*, Vol. 2 (1921), pp. 119-130.

‡ C. Kuratowski, *Fundamenta Mathematicae*, Vol. 3 (1922), pp. 59-64.

cyclic curve $* N$ of S such that $N \neq S$. If C is any component of $S - N$ then C has just one limit point, x , in N .^{*} If a and b are two points of N , then, since N is cyclicly connected, there is an arc t in N from a to b one of whose interior points is x .[†] If $y \in C$, the set $t + y$ is a set E , but it not a subset of any arc in S . Hence S is cyclicly connected.

Since S is a plane cyclicly connected continuous curve, the boundary of any domain complementary to S is either a simple closed curve or an open curve.[†] If $S \neq E_2$, there is at least one domain complementary to S .

Case 1. There is a domain D complementary to S whose boundary is a simple closed curve, J , of S .

It is easily shown that D is either the exterior or the interior of J . Let us suppose that D is the interior of J . (The argument is essentially the same if D is the exterior of J .) If $J \neq S$, there is a component C of $S - J$ which lies in the exterior of J . Since S is cyclicly connected, C has at least two limit points in J . Using the Wilder accessibility theorem, it follows that there exists an arc α whose end points a_1 and a_2 lie on J and which is such that $\alpha - a_1 - a_2 \subset C$. Let $a_1 a_2$ be that one of the two arcs of J from a_1 to a_2 which forms with α a simple closed curve whose interior lies exterior to J . Now let $r \in J - a_1 a_2$ and let a'_1 and a'_2 be points of $a_1 a_2$ such that $a_1 < a'_1 < a'_2 < a_2$. Then the E -set $\alpha + a_1 a'_1 + a_2 a'_2 + r$ is clearly a subset of no arc in S . Thus $J = S$.

Case 2. There is no domain complementary to S whose boundary is a simple closed curve. We shall show that, under our supposition that S contains a simple closed curve, this case is impossible.

Let D be any domain complementary to S . The boundary, B , of D is an open curve. Since S contains a simple closed curve, $B \neq S$. There is, then, a component C of $S - B$. In the same way as in Case 1, we prove the existence of an arc α whose end points, a_1 and a_2 , are points of B , and which is such that $\alpha - a_1 - a_2 \subset S - B$. Let $r \in B - a_1 a_2$ and let a'_1 and a'_2 be points of $a_1 a_2$ such that $a_1 < a'_1 < a'_2 < a_2$. As D must be identical with that domain of the plane complementary to B which does not contain $\alpha - a_1 - a_2$, it is clear that the E -set $\alpha + a_1 a'_1 + a'_2 a_2 + r$ is a subset of no arc in S . Thus, Case 2 has been shown to be impossible.

^{*} G. T. Whyburn, *Proceedings of the National Academy of Sciences*, Vol. 13 (1927), pp. 31-38.

[†] W. L. Ayres, *Bulletin de l'Académie Polonaise des Sciences et des Lettres* (1928); Theorem 3.

THEOREM 6. *If S is a closed set in a space E_n , and is of dimension 1 in at least one of its points, and if every bounded closed subset of S which satisfies the Moore-Kline conditions is a subset of an arc in S , then S is a simple continuous curve.*

Proof. As in the proof of the previous theorem, let us use the expression "a set E " to mean any bounded closed subset of S which satisfies the Moore-Kline conditions. As established in the previous theorem, S is a continuous curve, and if S contains no simple closed curve it is either an arc, a ray, or an open curve; and if S does contain a simple closed curve it is cyclicly connected.

We suppose, then, that S is cyclicly connected, and denote by p a point in which S is 1-dimensional. The point p lies on some simple closed curve J of S . Now if $J \neq S$, there is a component C of $S - J$. The component C has at least two limit points in J , since S is cyclicly connected. It follows that there is an arc in S which is a subset of $S - J$ except for one of its end points q which is a point of J distinct from p . Let Σ be a sphere whose center is p and which is so small that q does not lie in Σ or on its boundary. Since S is 1-dimensional in p there is a vicinity V of p such that $V \subset \Sigma$ and such that B , the boundary of V in S , is totally disconnected. As $B + p$ is a closed totally disconnected set and therefore a set E , it is a subset of an arc A of S . Now A cannot contain all the points of S outside V , for the point q lies outside V and is the junction point of three arcs which lie outside V . Therefore, there is a point h outside V which is not a point of A . If C_h is the component of $S - A$ which h determines, then C_h cannot have p as a limit point since p lies inside V , and h outside, and A contains the boundary B of V in S .

Now, there must be an arc in C_h from h to one of the end points a_1 or a_2 , say, a_1 —of A , for otherwise the E -set $A + h$ is a subset of no arc in S . Since S is cyclicly connected, C_h has another limit point in A . Let r be the last limit point of C_h on A in the order from a_1 to a_2 . The point r is arc-wise accessible from C_h . For let a_1r be the arc of A from a_1 to r , and let $\{r_n\}$ be a sequence of points in C_h such that $\lim r_n = r$. The set $a_1r + [r_n]$ is an E -set and therefore a subset of an arc in S . Clearly, this arc contains an arc from a point r_n to r which lies in C_h except for r . Thus r is accessible from C_h . Let t_1 and t_r be mutually exclusive arcs of S , where a_1 is an end point of t_1 , and r is an end point of t_r , and where $(t_1 - a_1) + (t_r - r) \subset C_h$. The point r must be a_2 , for suppose not. Then the point set $t_1 + a_1r + t_r + a_2$ is a set E , but is clearly not a subset of any arc in S . Hence $r = a_2$.

If there were a component C_w of $S - A$ distinct from C_h , the point set $t_1 + a_1r + t_r + w$ would be an E -set which is clearly not a subset of any arc in S . Thus C_h is the only component of $S - A$, so that $S = A + C_h$. Since a_1 and a_2 are both accessible from C_h , there is an arc L from a_1 to a_2 such that $L - a_1 - a_2 \subset C_h$. Now, p is an interior point of A and is not a limit point of C_h . Therefore there is a whole arc x_1px_2 of A , no point of which is a limit point of C_h , and where the order on A of the points a_1, x_1, p, x_2 and a_2 is a_1, x_1, p, x_2, a_2 . Let a_1x_1 and x_2a_2 be the arcs of A from a_1 to x_1 and from x_2 to a_2 , respectively. Suppose now that $C_h - C_h \cdot L \neq 0$ and let $s \in C_h - C_h \cdot L$. The point set $a_1x_1 + L + a_2x_2 + s$ is a set E , but is a subset of no arc in S . Hence $C_h = C_h \cdot L$ so that S is the simple closed curve $L + A$. We have shown, then, that if S contains a simple closed curve, S is a simple closed curve. This completes the proof of the theorem.

It is not apparent that in three dimensions the closed sets for which the Moore-Kline theorem is true fall into any easily classified category. For instance, besides the topological plane and sphere, and the simple continuous curves, the theorem is true in three dimensions for such sets as the cube together with its interior, any closed two-dimensional manifold, and various continua made up of combinations of closed manifolds.

We shall now consider the general problem of determining when a given closed subset M of a continuous curve S lies on an arc of S .

THEOREM 7. *If S is a bounded continuous curve, and M a closed subset of S which is not a subset of any one maximal cyclic curve of S , then M is a subset of an arc in S if and only if the following three conditions are satisfied:*

(1) *If C is a cyclic element* of S , at most two components of $S - C$ contain a point of M .*

(2) *If C is a maximal cyclic curve of S , and K_1 and K_2 are distinct components of $S - C$ such that $K_i \cdot M \neq 0$ ($i = 1, 2$), and such that a_1 , the unique limit point of K_1 in C is distinct from a_2 , the unique limit point of K_2 in C , then there is an arc t in C from a_1 to a_2 such that $M \cdot C \subset t$.*

(3) *If C is a maximal cyclic curve of S , and K is the only component of $S - C$ such that $K \cdot M \neq 0$, and a is the unique limit point of K in C , then there exists an arc t in C with a as one end point and such that $C \cdot M \subset t$.*

Proof. The conditions are sufficient.

* See G. T. Whyburn, *American Journal of Mathematics*, Vol. 50 (1928), pp. 167-194.

Let W be the arc-curve $S(M)$.* The set W is a continuous curve* consisting of a certain collection of cyclic elements of S , and the cyclic elements of W are cyclic elements of S .† It will first be shown that if C is a cyclic element of W , then $W - C$ contains at most two components. Let C be any cyclic element of W . If K is a component of $W - C$, then $K \cdot M \neq 0$. For suppose the contrary and let a be the unique limit point of K in C , and let $x \in K$. Since $K \subset W$, there are two points m_1 and m_2 of M which are the end points of an arc t in W which has x as an interior point. Not both of the sub-arcs m_1x and xm_2 of t can contain the point a . But m_1 and m_2 are both points of $W - K$ and a separates K from $W - K - a$. From this contradiction it follows that $K \cdot M \neq 0$.

Suppose, now, that $W - C$ contains three components K_i ($i = 1, 2, 3$). Each component K_i is a subset of a component K'_i of $S - C$. In virtue of (1), two of these components are identical. We may suppose $K'_1 = K'_2$. Let $x \in K_1 \cdot M$ and $y \in K_2 \cdot M$. Now, $x + y \subset K'_1$, and there is an arc t in K'_1 from x to y .‡ Since $t \subset W$ and $K_1 + t + K_2$ is a connected subset of $W - C$, K_1 and K_2 lie in one component of $W - C$. As this contradicts the definition of K_1 and K_2 , it follows that $W - C$ contains at most two components.

G. T. Whyburn has shown§ that any bounded continuous curve, H , which is not cyclicly connected contains at least two nodes, a node being either an end point of H or a maximal cyclic curve N of H such that $H - I(N)$ is connected, $I(N)$ being the set of all points of N which are not limit points of $H - N$.

Since, by our hypothesis, M is not a subset of any one maximal cyclic curve of S , and since the cyclic elements of W are all cyclic elements of S , W cannot be cyclicly connected. Accordingly let N be a node of W . Suppose $W - N$ is not connected. Then N is evidently not a point since no end point of a continuous curve cuts the continuous curve. We have $W - N = W_1 + W_2$, where W_1 and W_2 are mutually separated. As we have already shown, W_1 and W_2 are connected. The set W_1 has a unique limit point a_1 and W_2 a unique limit point a_2 in N . Now $a_1 \neq a_2$, for if $a_1 = a_2$, the set $W - a_1$ contains three components $N - a_1$, W_1 and W_2 . Evidently $W - I(N)$

* See W. L. Ayres, *Transactions of the American Mathematical Society*, Vol. 30 (1928), pp. 567-578.

† W. L. Ayres, *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 595-612.

‡ See, for instance, R. L. Moore, *Bulletin of the American Mathematical Society*, Vol. 23 (1917), pp. 233-236.

§ See G. T. Whyburn, *American Journal of Mathematics*, loc. cit.

$= (W_1 + a_1) + (W_2 + a_2)$. Since W_1 and W_2 are mutually separated and since $a_1 \cdot \bar{W}_2 = a_2 \cdot \bar{W}_1 = 0$ it follows that $(W_1 + a_1)$ and $(W_2 + a_2)$ are mutually separated. As this contradicts the fact that N is a node, every node of W must be a non-cut element of W , so that W contains at least two non-cut elements N_1 and N_2 .

Let X be any simple cyclic chain* in W from N_1 to N_2 . It will be shown that $X = W$. Suppose, in fact, $X \neq W$, and let $p \in W - X$. The point p determines a component C_p of $W - X$ which has a unique limit point q in X . Since N_1 and N_2 are non-cut elements of W , they are, in virtue of a theorem of Whyburn's,† end elements of W . In other words, q is not a point of either N_1 or N_2 . Hence, q is a point of some cyclic element Q of X interior to X . Clearly, $W - Q$ contains three components determined by $N_1 - N_1 \cdot Q$, $N_2 - N_2 \cdot Q$, and C_p . As we have shown that this is impossible, it follows that W is the simple cyclic chain X .

Now, let t be an arc in W from a point of N_1 to a point of N_2 . In virtue of a theorem of Whyburn's,‡ the arc t contains a point of every cyclic element of W , and in fact, if C is a maximal cyclic curve of W interior to X , then $t \cdot C$ is a sub-arc of t whose end points are the unique limit points in C of the two distinct components of $W - C$ and therefore of $S - C$. Let us now replace all such sub-arcs of t by the corresponding arcs whose existence is given by (2), and let us replace $t \cdot N_1$ and $t \cdot N_2$ by sub-arcs of N_1 and N_2 of the sort whose existence is given by (3). Since for every $\epsilon > 0$ there are only a finite number of maximal cyclic curves of W of diameter $> \epsilon$ it is readily shown that the resulting point set is an arc, A . As $M \subset A \subset W \subset S$, the conditions have been shown to be sufficient.

The conditions are also necessary.

Let t be an arc of S such that $t \supset M$, and let us suppose that there is a cyclic element C of S such that three components, S_1 , S_2 , and S_3 of $S - C$ each contain a point of M . Let $p_i \in M \cdot S_i$, and let us suppose that the order of the points p_1 , p_2 , and p_3 on t is p_1, p_2, p_3 . Since these three points lie in different components of $S - C$, the sub-arcs $p_1 p_2$ and $p_2 p_3$ of t each contain a point of C . Evidently, then, S_2 has at least two limit points in C . As this is impossible, the necessity of (1) is proved.

To prove the necessity of (2) let C be any maximal cyclic curve of S and suppose there exist components K_1 and K_2 of $S - C$ and points a_1 and a_2

* See G. T. Whyburn, *loc. cit.*

† *Loc. cit.*

‡ *Loc. cit.*

as specified in (2). Since each point a_1 and a_2 separates K_1 from K_2 , t must contain a_1 and a_2 . The sub-arc a_1a_2 of t is a subset of C , for otherwise there is determined a component of $S - C$ which has at least two limit points in C . If $(t - a_1a_2) \cdot C \neq 0$ either K_1 or K_2 would have two limit points in C . Hence $a_1a_2 = t \cdot C$. Since $t \supset M \supset C \cdot M$, the arc a_1a_2 of C must contain $C \cdot M$. The necessity of (3) follows in a similar way. This completes the proof of the theorem.

We shall now make certain applications of the theorem just proved.

THEOREM 8. *If the plane continuous curve S is the boundary of a (connected) domain D and M is any closed subset of S other than a simple closed curve, then M is a subset of an arc in S if and only if the following three conditions are satisfied:*

(1) *If F is a point or a simple closed curve of S , at most two components of $S - F$ contain a point of M .*

(2) *If J is a simple closed curve in S , and if there are two components of $S - J$ which contain a point of M and have distinct limit points a_1 and a_2 in J , then only one of the arcs of J from a_1 to a_2 contains a point of M .*

(3) *If J is a simple closed curve in S , and if just one component of $S - J$ contains a point of M , then the (unique) limit point in J of this component is not a limit point of $J \cdot M$ from both directions along J .**

Proof. The theorem is almost an immediate consequence of Theorem 7 and the fact that every maximal cyclic curve of a continuous curve which is the boundary of a plane domain is a simple closed curve.† It is clear that the above theorem is true for any continuous curve in E_n whose maximal cyclic curves are all simple closed curves.

THEOREM 9. *If S is an acyclic continuous curve in E_n and M is a closed subset of S , then in order that M be a subset of an arc in S , it is necessary and sufficient that if p is a point of S , at most two components of $S - p$ contain a point of M .*

Proof. This theorem is an immediate consequence of Theorem 7 and the fact that every cyclic element of an acyclic continuous curve is a point.

Theorem 9 can evidently be made more general by letting S be any continuous curve in E_n , M any closed subset of S which has no point on any simple closed curve of S , and p any cyclic element of S .

* The phrase "from both directions along J " is immediately translatable into topological language.

† R. L. Wilder, *Fundamenta Mathematicae*, Vol. 7 (1923), pp. 340-377.

With a view to making another application of Theorem 7, the following theorem will be proved:

THEOREM 10. *If J is a simple closed curve in E_2 , and I is its interior, and if F is a closed totally disconnected subset of $J + I$, then there is an arc in $J + I$ which contains F .*

Proof. Let us denote $F \cdot J$ by T . There is an arc AB of J which has T in its interior. Let C and D be two other points of J and let the order of the points A, B, C , and D on J be A, B, C, D, A , where T is contained in the arc ABC of J . Let a system of rectangular coördinates be given, and let A', B', C' and D' be the points $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. Let J' be the simple closed curve formed by the square $A'B'C'D'A'$, and let I' be its interior. There will be defined a $1-1$ bicontinuous mapping of $J + I$ into $J' + I'$ which has the properties:

- (1) J corresponds to J' ; the points of AB corresponding to the points of $A'B'$, A corresponding to A' and B to B' .
- (2) The points of T correspond to irrational points of $A'B'$.
- (3) The points of $F - T$ correspond to points of I' both of whose coördinates are irrational.

The proof that such a mapping exists is based on a lemma in a paper of R. L. Wilder's,* and for the meaning of the notation used in the next paragraph, the reader is referred to the paper in question.

There is a $1-1$ bicontinuous correspondence between the points of AB and the points of $A'B'$ in which A corresponds to A' and B to B' , the rational points within $A'B'$ corresponding to points of AB which are end points of the arcs a_1 , and the irrational points of $A'B'$ corresponding to points of AB which are end points of the arcs b_1 . There is a like $1-1$ bicontinuous correspondence between the points of $A'D'$ and AD . These two correspondences induce like correspondences between the points of BC and $B'C'$, CD and $C'D'$. If $p \in I$, there is just one arc t_1 of α_1 , and just one arc t_2 of α_2 which contains p . That end point of t_1 which belongs to AB corresponds to a point x of $A'B'$, and that end point of t_2 which belongs to AD corresponds to a point y of $A'D'$. To the point p we let correspond the point (x, y) of I' . It is readily shown that the correspondence in question is $1-1$ bicontinuous and has the properties (1), (2), and (3) specified above.

Consider, now, the square J' , and let F' and T' be the sets corresponding

* *Transactions of the American Mathematical Society*, Vol. 31 (1929), pp. 345-359; Lemma 6, p. 356.

to F and T . The set T' is a closed non-dense subset of $A'B'$. Let R_1 be the rectangle whose vertices are the points $(0, \frac{1}{2})$, $(1, \frac{1}{2})$, $(1, 1)$ and $(0, 1)$. The projection, p_1 , on $A'B'$ of the mid-point of the base of R_1 is the point $(\frac{1}{2}, 0)$. Since p_1 is a rational point, it belongs to some sub-interval of $A'B'$ complementary to T' . Let $(a_1, 0)$ and $(b_1, 0)$ be any two rational points of this interval such that $a_1 < \text{absc. of } p_1 < b_1$. Let J_1 be the simple closed curve obtained by deleting from R_1 the sub-interval of R_1 between $(a_1, \frac{1}{2})$ and $(b_1, \frac{1}{2})$ and adding the lines representing the ordinates of these two points and the interval, i_1 , of $A'B'$ from $(a_1, 0)$ to $(b_1, 0)$. Denote, now, by R_2 the rectangle whose vertices are the points $(0, \frac{1}{2})$, $(a_1, \frac{1}{2})$, $(a_1, \frac{1}{2})$ and $(0, \frac{1}{2})$. The projection, p_2 , on $A'B'$ of the mid-point of the base of R_2 is a rational point and therefore belongs to some sub-interval of $A'B'$ complementary to T' . Let $(a_2, 0)$ and $(b_2, 0)$ be any two rational points of this interval such that $a_2 < \text{absc. of } p_2 < b_2$, and let us denote the interval of $A'B'$ from $(a_2, 0)$ to $(b_2, 0)$ by i_2 . These points are related to R_2 in the same way that $(a_1, 0)$ and $(b_1, 0)$ are related to R_1 . Let us, then, construct J_2 in relation to them and to R_2 in the same way that J_1 was constructed in relation to $(a_1, 0)$ and $(b_1, 0)$ and R_1 . We now consider the rectangle R_3 whose vertices are the points $(b_1, \frac{1}{2})$, $(1, \frac{1}{2})$, $(1, \frac{1}{2})$ and $(b_1, \frac{1}{2})$. We define in a similar way a simple closed curve J_3 . At the next stage of the process we have 2^2 rectangles to consider, then 2^3 rectangles, and so on.

Proceeding indefinitely in the manner indicated, it is clear that we obtain a sequence $\{J_n\}$ of simple closed curves such that:

- (1) $J_n \cdot F' = 0 \quad (n = 1, 2, \dots)$.
- (2) If H_n is the interior of J_n , then $H_i \cdot H_j = 0 \quad (i \neq j) \quad (i, j = 1, 2, \dots)$.
- (3) $F' - T' \subset \sum_{n=1}^{\infty} H_n$.
- (4) $\lim \text{diam}(J_n) = 0$.

The set F'_n is a closed totally disconnected set. There is, then, in virtue of Theorem 2 of this paper an arc t_n in H_n such that $t_n \supset F'_n$. It is easy to show that the end points of this arc can be joined by arcs to a_n and b_n so as to give an arc t'_n such that $t'_n \supset F'_n$ and $t'_n - a_n - b_n \subset H_n$. It follows easily that the point set $\sum_{n=1}^{\infty} t'_n + (A'B' - \sum_{n=1}^{\infty} i_n)$ is an arc, t' , whose end points are A' and B' . Evidently $F' \subset t' \subset J' + I'$. In virtue of the homeomorphism between $J + I$ and $J' + I'$, the theorem is established.

THEOREM 11. *Under the conditions of Theorem 10, if a and b are two*

points of F , then there is an arc in $J + I$ which contains F and has a and b as its end points.

The proof is based on Theorem 10 and uses the homeomorphism between $J + I$ and $J' + I'$ set up in the proof of that theorem.

It is possible to show that the arc of Theorem 11 can be so constructed that the only points it has in common with J are the points of F in J .

We shall now use Theorem 11 to make another application of Theorem 7.

THEOREM 12. *If S is a bounded plane continuous curve which does not cut the plane, and F is a closed totally disconnected subset of S , then in order that F lie on an arc of S , it is necessary and sufficient that for every cyclic element C of S at most two components of $S - C$ contain a point of F .*

Proof. Every maximal cyclic curve of S is a simple closed curve plus its interior.* If C is such that there are two points a_1 and a_2 of the sort specified under condition (2), Theorem 7, then in virtue of Theorem 11, there is an arc in C which contains the closed totally disconnected set $C \cdot F + a_1 + a_2$ and has a_1 and a_2 as its end points. Similarly, if condition (3), Theorem 7, applies, there is an arc in C which contains $C \cdot F + a$, and has a as one of its end points. Thus the conditions of Theorem 7 are all satisfied and Theorem 12 follows.

As a special case of the general problem of this paper it is of interest to consider the problem of determining necessary and sufficient conditions that a given finite subset of a continuous curve S lie on an arc of S . A condition which is sufficient but not necessary has been given by W. L. Ayres.† Evidently, if S is any set of points whatsoever and M is a subset of S consisting of just n points, then for M to lie on an arc of S it is necessary that if $p_1, \dots, p_i$ are any i points of S ($i < n - 1$) at most $i + 1$ components of $S - (p_1 + \dots + p_i)$ contain a point of M . This condition is clearly not sufficient for continuous curves in general.

We have found conditions which are at the same time necessary and sufficient that a given set of n points of a continuous curve S lie on an arc of S for the cases $n = 3$ and $n = 4$. These are embodied in the following two theorems whose proofs we do not feel it will be necessary to include.

THEOREM 13. *If S is a continuous curve in E_n and M is a subset of S consisting of just three points, then M is a subset of an arc in S if and only*

* See G. T. Whyburn, *American Journal of Mathematics*, loc. cit.

† This paper is not yet published. For abstract, see *Bulletin of the American Mathematical Society*, Vol. 35 (1929), p. 772.

if M contains a point which is not separated from both of the other points of M by any one point of S .

THEOREM 14.* *If S is a cyclicly connected continuous curve in E_n and M consists of four points of S , then M is a subset of an arc in S if and only if for every pair of points p_1 and p_2 of S at most three components of $S - (p_1 + p_2)$ contain a point of M .*

Simple examples show that none of the evident generalizations of the conditions in Theorems 13 and 14 apply to the general case of *any* finite subset of a continuous curve.

* This theorem is a generalization of a theorem of W. L. Ayres, *Bulletin de l'Académie Polonaise des Sciences et des Lettres* (1928), Theorem 5.

